Changing Measures.

Let Z be a non-negative random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}(Z) = 1$. The function

$$\mathcal{F} \ni E \xmapsto{\mathbb{Q}} \int_E Z \, d\mathbb{P}$$

is a probability measure on $\mathcal{F}$, this follows from the Radon-Nikodym Theorem*.

*Or, (i) $\mathbb{Q}(\Omega) = \int_\Omega Z \, d\mathbb{P} = \mathbb{E}(Z) = 1$

(ii) If $E_n \in \mathcal{F}$ and are disjoint with $E = \bigcup_n E_n$ then,

$$\mathbb{Q}\left(\bigcup_n E_n\right) = \int_{\bigcup_n E_n} Z \, d\mathbb{P} = \int_\Omega Z I_{\bigcup_n E_n} \, d\mathbb{P} = \int_\Omega Z \sum_n I_{E_n} \, d\mathbb{P}$$

$$= \sum_n \int_\Omega Z I_{E_n} \, d\mathbb{P} \quad \text{(Monotone Convergence)}$$

$$= \sum_n \mathbb{Q}(E_n)$$

It could be that $\mathbb{Q}(E) = 0$ while $\mathbb{P}(E) > 0$. Certainly, if $\mathbb{P}(E) = 0$ then $\mathbb{Q}(E) = 0$ — integration theory! We say $\mathbb{Q}$ is continuous with respect to $\mathbb{P}$. If $Z > 0$ $\mathbb{P}$ almost surely then $\mathbb{Q}(E) = 0 \Rightarrow \mathbb{P}(E) = 0$, again by integration theory. When we have $\mathbb{P}(E) = 0 \Leftrightarrow \mathbb{Q}(E) = 0$ we say $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, $\mathbb{Q} \sim \mathbb{P}$.

Conditional Expectation and Changes of Measure.

The conditional expectation is characterised by two conditions. Let M_q be the

Conditional expectation of $L^1(\Omega, \mathcal{F}, \mathbb{P})$ onto $L^1(\Omega, \mathcal{G}, \mathbb{P})$.

(i) For $f \in L^1(\mathcal{F})$, $M_{\mathcal{G}}(f)$ is $\mathcal{G}$-measurable,

(ii) $\forall H \in \mathcal{G}$, $\displaystyle\int_\Omega I_H f \, d\mathbb{P} = \int_\Omega I_H M_{\mathcal{G}}(f) \, d\mathbb{P}$.

So $M_{\mathcal{G}}$ is $\mathbb{P}$ dependant! We should, and will, write $M^{\mathbb{P}}_{\mathcal{G}}$.

Question: What is the relationship between $M^{\mathbb{Q}}_{\mathcal{G}}$ and $M^{\mathbb{P}}_{\mathcal{G}}$ when $\mathbb{Q} \sim \mathbb{P}$?

We have $\mathbb{Q}(E) = \int_E Z \, d\mathbb{P}$ and it is not difficut to prove that

$$\int_\Omega X \, d\mathbb{Q} = \int_\Omega X Z \, d\mathbb{P} .$$

For each $H \in \mathcal{G}$,

$$\int_H X \, d\mathbb{Q} = \int_H M^{\mathbb{Q}}_{\mathcal{G}}(X) \, d\mathbb{Q} = \int_H M^{\mathbb{Q}}_{\mathcal{G}}(X) Z \, d\mathbb{P}$$

$$= \int_H M^{\mathbb{P}}_{\mathcal{G}}\left(M^{\mathbb{Q}}_{\mathcal{G}}(X) Z\right) \, d\mathbb{P}$$

$$= \int_H M^{\mathbb{Q}}_{\mathcal{G}}(X) M^{\mathbb{P}}_{\mathcal{G}}(Z) \, d\mathbb{P}$$

and

$$\int_H X \, d\mathbb{Q} = \int_H X Z \, d\mathbb{P} = \int_H M^{\mathbb{P}}_{\mathcal{G}}(XZ) \, d\mathbb{P}$$

which is $\displaystyle = \int_H M^{\mathbb{Q}}_{\mathcal{G}}(X) M^{\mathbb{P}}_{\mathcal{G}}(Z) \, d\mathbb{P}$

So $M_G^{\mathbb{P}}(XZ) = M_G^{\mathbb{Q}}(X) M_G^{\mathbb{P}}(Z)$ (as $H \in G$)

that is, $$M_G^{\mathbb{Q}}(X) = \frac{M_G^{\mathbb{P}}(XZ)}{M_G^{\mathbb{P}}(Z)}.$$

Martingales under $\mathbb{P}$ and $\mathbb{Q}$. So now our condex are indexed by time, t. We have,
$$M_t^{\mathbb{Q}}(X) = \frac{M_t^{\mathbb{P}}(XZ)}{M_t^{\mathbb{P}}(Z)}$$

Let $Z_t = M_t^{\mathbb{P}}(Z)$.

Theorem

(Y_t) is a $\mathbb{Q}$ martingale iff $(Y_t Z_t)$ is a $\mathbb{P}$-martingale.

Pf

If (Y_t) is a $\mathbb{Q}$-martingale then $M_s^{\mathbb{Q}}(Y_t) = Y_s$ for $s \leq t$. That is,
$$M_s^{\mathbb{Q}}(Y_t) = \frac{M_s^{\mathbb{P}}(Y_t Z)}{Z_s} = Y_s.$$

This states that $M_s^{\mathbb{P}}(Y_t Z) = Y_s Z_s$. But
$$M_s^{\mathbb{P}}(Y_t Z) = M_s^{\mathbb{P}} \circ M_t^{\mathbb{P}}(Y_t Z) = M_s^{\mathbb{P}}(Y_t M_t^{\mathbb{P}}(Z))$$
$$= M_s^{\mathbb{P}}(Y_t Z_t).$$

So $(Y_t Z_t)$ is a martingale. Conversely;

If $(Y_t Z_t)$ is a $\mathbb{P}$-martingale then $M_s^{\mathbb{P}}(Y_t Z_t) = Y_s Z_s$ and

$$Y_s = \frac{M_s^{\mathbb{P}}(Y_t Z_t)}{Z_s} = \frac{M_s^{\mathbb{P}}(Y_t M_t^{\mathbb{P}}(Z))}{Z_s}$$

$$= \frac{M_s^{\mathbb{P}}(M_t^{\mathbb{P}}(Y_t Z))}{Z_s}$$

$$= \frac{M_s^{\mathbb{P}}(Y_t Z)}{Z_s}$$

$$= M_s^{\mathbb{Q}}(Y_t)$$

So (Y_t) is a $\mathbb{Q}$ mart.

These results are very important !!!